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# Aspects of the Noisy Burgers Equation

Hans Fogedby

[1] Institute of Physics and Astronomy, University of Aarhus, DK-8000, Aarhus C
[2] NORDITA, Blegdamsvej 17, DK-2100, Copenhagen Ø, Denmark

**Abstract.** The noisy Burgers equation describing for example the growth of an interface subject to noise is one of the simplest model governing an intrinsically nonequilibrium problem. In one dimension this equation is analyzed by means of the Martin-Siggia-Rose technique. In a canonical formulation the morphology and scaling behavior are accessed by a principle of least action in the weak noise limit. The growth morphology is characterized by a dilute gas of nonlinear soliton modes with gapless dispersion law $E \propto p^{3/2}$ and a superposed gas of diffusive modes with a gap. The scaling exponents and a heuristic expression for the scaling function follow from a spectral representation.

## 1 Introduction

Macroscopic phenomena far from equilibrium are ubiquitous and include phenomena such as turbulence in fluids, interface and growth problems, chemical reactions, biological processes, and even aspects of economical and sociological structures.

In recent years much of the focus of modern statistical physics and soft condensed matter has shifted towards such systems. Drawing on the case of static and dynamic critical phenomena in and close to equilibrium, where scaling, critical exponents, and the concept of universality have so successfully served to organize our understanding and to provide a variety of calculational tools, a similar approach has been advanced towards the much larger class of nonequilibrium phenomena with the purpose of elucidating scaling properties and more generally the morphology or pattern formation in a driven state.

In this context the noisy Burgers equation in one dimension provides maybe the simplest continuum description of an open driven nonlinear system exhibiting both scaling and pattern formation. This equation has the form (Forster et al. (1976), Forster et al. (1977))

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u + \lambda u \nabla u + \nabla \eta \ , \tag{1}$$

and was in the noiseless case for $\eta = 0$ originally proposed by Burgers (Burgers (1974)) in order to model turbulence in fluids; we note the similarity with the Navier-Stokes equation for $\lambda = -1$. Equation (1) has the form of a conserved nonlinear Langevin equation, $\partial u/\partial t = -\nabla j$, with fluctuating current $j = -\nu \nabla u - (\lambda/2)u^2 + \eta$. The linear diffusive damping term $\nu \nabla^2 u$ is characterized by the surface tension or viscosity $\nu$. The nonlinear convective or mode

coupling term $\lambda u \nabla u$ is controlled by $\lambda$. Finally, the equation is driven by the fluctuating "white noise" $\eta$ for which we assume a Gaussian amplitude distribution

$$P(\eta) = \exp\left[-\frac{1}{2\Delta}\int dxdt\eta(xt)^2\right] \ , \tag{2}$$

and short-range correlations in space according to the correlation function

$$\langle \eta(xt)\eta(x't')\rangle = \Delta\delta(x-x')\delta(t-t') \ , \tag{3}$$

characterized by the noise strength parameter $\Delta$.

In the context of modeling a growing interface the Burgers equation governs the local slope $u = \nabla h$ of a height field $h$ (in the Monge representation) characterized by the much studied Kardar-Parisi-Zhang equation (Kardar et al.(1986), Medina et al. (1989))

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta \ . \tag{4}$$

In this case, which we shall adhere to in the following, $\nu$ is a diffusion coefficient or viscosity, $\lambda$ a nonlinear lateral growth parameter, and $\eta$ represents noise in the drive or the environments. In Fig. 1 we have sketched the growth morphology in terms of the height and slope fields for a typical growing interface.

The substantial conceptual problems encountered in nonequilibrium physics are in many ways embodied in the Burgers-KPZ equations (1) and (4) describing the self-affine growth of an interface subject to annealed noise arising from fluctuations in the drive or in the environment. Interestingly, the Burgers-KPZ equations are also encountered in a variety of other problems such as randomly stirred fluids, dissipative transport in a driven lattice gas, the propagation of flame fronts, the sine-Gordon equation, and magnetic flux lines in superconductors. Furthermore, by means of the Cole- Hopf transformation the Burgers-KPZ equations are also related to the problem of a directed polymer or a quantum particle in a random medium and thus to the theory of spin glasses; see e.g. (Halpin-Healy and Zhang (1995)).

One issue which has been addressed is the scaling properties of the noisy Burgers equation. In Fig. 2 we have depicted how the width of the fluctuating interface (see Fig. 1) after a transient lapse of time, characterized by the a crossover time $t_{crossover}$ scaling with the system size $L$ according to $t_{crossover} \propto L^z$, where $z$ is the dynamic exponent, saturates to a value $w_0 \propto L^\zeta$ also depending on $L$ and characterized by the roughness exponent $\zeta$.

This dynamical scaling hypothesis, which is substantiated by numerical modeling and renormalization group considerations, is embodied in the following relationship for the width:

$$w(t,L) = L^\zeta F_1(t/L^z) \ . \tag{5}$$

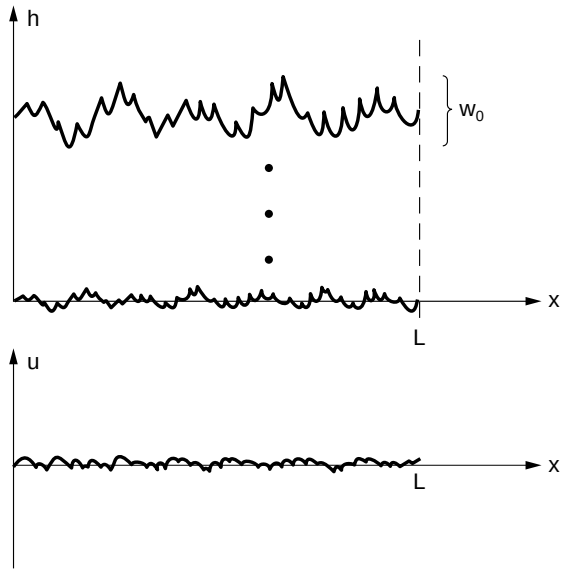


**Fig. 1.** We depict the growth morphology in terms of the height ($h$) and slope ($u$) fields for a growing interface. The saturated width in the stationary state is denoted by $w_0$.

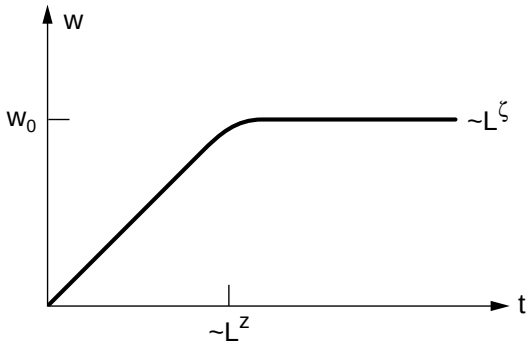


**Fig. 2.** We depict the interface width $w(t)$ as a function of time $t$. In the transient regime $t \ll t_{crossover} \sim L^{\zeta}$ $w$ grows according to $t^{\zeta/z}$. In the stationary regime attained for $t \gg t_{crossover}$ $w$ saturates to the value $w_0 \sim L^{\zeta}$.

In terms of the height field itself or the associated slope field the appropriate dynamical scaling forms are given by

$$\langle [h(x+x_0,t+t_0)-h(x_0,t_0)]^2\rangle = x^{2\zeta}F_2(t/x^z) \ , \tag{6}$$

and

$$\langle u(x+x_0,t+t_0)u(x_0,t_0)\rangle = x^{2\zeta-2}F_3(t/x^z) \ . \tag{7}$$

Here $x_0$ and $t_0$ are reference points in an infinite system in the stationary regime and $F_i$ the scaling functions. The scaling issue thus amounts to a determination of the scaling exponents $\zeta$ and $z$ together with the scaling functions $F_i$.

The hydrodynamical origin of the Burgers equation, as reflected by the presence of the mode coupling or convective term $\lambda u\nabla u$, implies that the Burgers equation is invariant under a Galilean transformation

$$x \ \rightarrow x-\lambda u_0 t \tag{8}$$

$$u \ \rightarrow u+u_0 \ , \tag{9}$$

involving a shift of the slope field. Since the nonlinear coupling strength $\lambda$ enters as a structural constant in the symmetry group it is invariant under scaling. This property implies a dynamical scaling law

$$\zeta + z = 2 \ , \tag{10}$$

relating the roughness and dynamic exponents.

Another interesting property of the Burgers equation specific to one dimension is the existence of an effective static fluctuation-dissipation theorem, in the sense that the stationary Fokker-Planck equation for the Burgers equation is solved by a Gaussian distribution

$$P_{\rm st}(u) \propto \exp\left[-\frac{\nu}{\Delta}\int dx u(x)^2\right] \tag{11}$$

for the slope field *independent* of the nonlinear coupling strength $\lambda$. This implies that the slope field $u$ performs independent Gaussian fluctuations and that, consequently, the height field $h=\int u dx$, performs random walk in $x$, yielding the roughness exponent and from the scaling law (10) also the dynamic exponent. Summarizing, the universality class for the Burgers equation is characterized by the exponents:

$$\zeta = \frac{1}{2} \qquad \text{and} \qquad z = \frac{3}{2} \ . \tag{12}$$

By a combination of the Galilean invariance and the effective fluctuation-dissipation theorem specific to one dimension the exponents in one dimension are thus exactly known. The scaling function has also been accessed numerically and by a mode coupling approach. In higher dimensions the scaling properties of the Burgers-KPZ are controversial and remain to be clarified. In this context the

dynamic renormalization group based on an expansion in $\lambda^2\Delta/\nu^3$ and an expansion about the critical dimension $d = 2$ yields limited results and in particular fails to access the strong coupling fixed point in $d = 1$, characterized by the exponents in (12).

In a recent series of papers (Fogedby et al.(1995), Fogedby (1998)) we have presented a novel approach to the strong coupling features of the noisy Burgers equation in one dimension based on a nonperturbative approach in the asymptotic weak noise limit. Since the singular character of the weak noise limit is an essential feature of our approach and is already apparent in the linear case for $\lambda = 0$ it is instructive first to consider this case.

## 2 The Edwards – Wilkinson Equation

In the linear case for $\lambda = 0$ the Burgers equation (1) takes the form of the Edwards-Wilkinson (EW) equation (for the slope field) (Edwards and Wilkinson (1982))

$$\partial u/\partial t = \nu\nabla^2 u + \nabla\eta \ , \tag{13}$$

i.e., a linear diffusion equation driven by conserved noise.

Equation (13) is readily analyzed. The time-dependent probability distribution for the wavenumber modes $u_k = \int dx u(x) \exp(-ikx)$, $u_k^* = u_{-k}$, is given by

$$P(\{u_k\}, t|\{u_k^0\}) \propto \prod_k \exp\left[-\frac{\nu}{\Delta}\frac{1}{L}\frac{|u_k - u_k^0 e^{-\nu k^2 t}|^2}{1 - e^{-2\nu k^2 t}}\right] \ , \tag{14}$$

where $u_k^0$ is the initial value for $t = 0$. For $t \to \infty$ $P(\{u_k\}, t)$ approaches the stationary distribution (11), using $(1/L)\sum_k |u_k|^2 = \int dx u(x)^2$.

In a similar way we obtain for the slope correlations in $\omega k$-space, $u(k\omega) = \int dx dt \exp(i\omega t - ikx)$, the Lorentzian line shape characteristic of a conserved diffusive mode,

$$\langle u(k\omega)u(-k-\omega)\rangle = \frac{\Delta k^2}{\omega^2 + (\nu k^2)^2} \ , \tag{15}$$

implying the static correlations $\langle u(x)u(x')\rangle = (\Delta/2\nu)\delta(x - x')$ in conformity with (11). In Fig. 3 we have depicted the correlation function.

Comparing (15) with the scaling form in (7) we also infer the scaling exponents $\zeta = 1/2$ and $z = 2$, characteristic of diffusion and defining the EW universality class. Also, noting that the diffusive term in (13) can be derived from a free energy $F = (1/2)\int dx u^2$, it follows that the EW equation describes the fluctuations in an equilibrium system with "temperature" $\Delta/2\nu$, i.e., $P_{\rm st} = \exp[-(2\nu/\Delta)F]$.

We note already here that the noise strength $\Delta$ plays a special role. Whereas $\Delta$ enters linearly in the correlation function (15), the limit $\Delta \to 0$ appears as

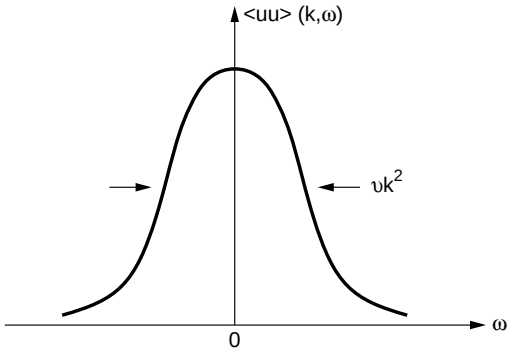


**Fig. 3.** We depict the slope correlation function for the diffusive mode in the EW case. The Lorentzian is centered about $\omega = 0$ and has the hydrodynamical linewidth $\nu k^2$ vanishing in the long wave length limit.

an "essential singularity" in the distribution functions $P(\{u_k\}, t)$ and $P_{\rm st}(u)$ in (11) and (14). The special role of the noise as the relevant small parameter is more physically recognized by considering the time-dependent correlations

$$\langle u(kt)u(-kt')\rangle = \\ [\langle uu\rangle_i - (\Delta/2\nu)]\exp[-(t+t')\nu k^2] + (\Delta/2\nu)\exp[-|t-t'|\nu k^2] \ , \quad (16)$$

$\langle\cdots\rangle_i$ denotes an initial value average. The basic time scale is set by $\tau(k) = 1/\nu k^2$ which diverges for $k \to 0$, characteristic of a hydrodynamical mode. However, there is another "time scale" set by the noise, namely a characteristic crossover time

$$\tau(\Delta) \sim \tau(k)\ln\frac{1}{\Delta} \ . \quad (17)$$

For $t, t' \ll \tau(k)$ the correlations are nonstationary and depend on the initial correlation $\langle\cdots\rangle_i$, whereas for long times $t, t' \gg \tau(k)$ the correlations enter a stationary, time reversal invariant regime and depends on $|t-t'|$. The crossover time $\tau(\Delta)$ defines the onset of the stationary regime. For $t, t' \gg \tau(k), \tau(\Delta)$ noise-induced fluctuations built up and the system becomes stationary; for $\Delta \to 0$, $\tau(\Delta) \to \infty$, and the system never leaves the transient regime, i.e., there is no stationary regime.

Whereas the singular nature of the weak noise limit is easily understood in the context of the linear EW equation; it is in fact precisely equivalent to the low temperature limit of the Boltzmann factor $\exp(-F/T), T = \Delta/2\nu$, the situation

is more subtle in the Burgers equation where the nonlinear growth term leads to a general nonequilibrium system.

In order to understand the role of the nonlinear term we first consider the well-understood properties of the noiseless or deterministic Burgers equation.

## 3 The Noiseless Burgers Equation

The noiseless Burgers equation has the form

$$\partial u/\partial t = \nu\nabla^2 u + \lambda u\nabla u \ , \tag{18}$$

and describes the transient decay of an interface subject to the damping term $\nu\nabla^2 u$ in combination with the mode coupling or nonlinear growth term $\lambda u\nabla u$. Correspondingly, for the height field we obtain the deterministic KPZ equation

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 \ . \tag{19}$$

Interestingly, the noiseless Burgers equation (18) can be solved analytically by means of the nonlinear Cole-Hopf transformation (Cole (1951) and Hopf (1950))

$$w = \ \exp\left[\frac{\lambda}{2\nu}\int dx u\right] \ , \tag{20}$$

which maps (18) onto the linear diffusion equation

$$\frac{\partial w}{\partial t} = \nu\nabla^2 w \ , \tag{21}$$

and an initial value analysis can be carried out, in particular in the inviscid limit $\nu \to 0$. The basic transient mode structure consists of i) solitons or shock waves, connected by ii) ramp modes, and iii) superposed diffusive modes.

### 3.1 Linear Modes for $\lambda = 0$

In the linear case for $\lambda = 0$ the noiseless Burgers equation reduces to the ordinary diffusion equation

$$\frac{\partial u}{\partial t} = \nu\nabla^2 u \ , \tag{22}$$

which supports decaying diffusive modes

$$u(xt) \propto \exp\left[-i\omega_k^0 t + ikx\right] \ , \tag{23}$$

with gapless quadratic dispersion

$$\omega_k^0 = -i\nu k^2 \ . \tag{24}$$

In Fig. 4 we have depicted the diffusive spectrum

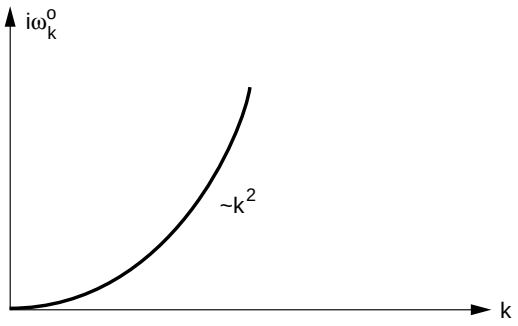


**Fig. 4.** We show the gapless dispersion law for the diffusive modes for the EW case.

### 3.2 Soliton and Ramp Modes for $\lambda \neq 0$

In the presence of the nonlinear mode coupling term the noiseless Burgers equation, violating parity symmetry, supports a localized "right hand" soliton mode of the sine-Gordon form

$$u_k(x) = u_+ \tanh k_s(x - x_0) \ , \tag{25}$$

with center position $x_0$ and amplitude $u_+$. The inverse width

$$k_s = \frac{\lambda u_+}{2\nu} \ , \tag{26}$$

depends on the amplitude; in the inviscid limit $\nu \to 0$, $k_s \to \infty$, and the soliton becomes a sharp shock wave. Owing to the Galilean invariance in (8-9) we also easily obtain a boosted soliton moving with velocity $v$. Denoting the boundary values at $x = \pm\infty$ by $u_\pm$, we infer the single soliton condition

$$u_+ + u_- = -\frac{2v}{\lambda} \ , \tag{27}$$

expressing the soliton velocity in terms of the boundary values $u_\pm$. The height profile corresponding to a soliton has the form

$$h(xt) = \frac{2\nu}{\lambda} \ln \cosh \left[k_s(x - x_0)\right] \ . \tag{28}$$

The soliton mode is a stable "dissipative structure" corresponding to deterministic input at the boundaries (nonvanishing currents) and dissipation at the soliton

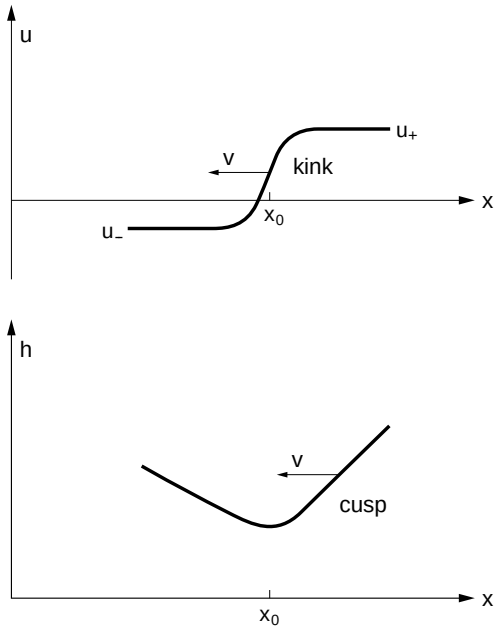


**Fig. 5.** We depict a moving "right hand" soliton for the ' noiseless Burgers equation together with the associated downward cusp in the height profile.

center (hot spot). In Fig. 5 we have depicted a moving soliton and the associated height profile.

Furthermore, in the inviscid limit $\nu \to 0$ (18) supports a negative-slope ramp solution

$$u(xt) \propto -\frac{x}{\lambda t} \ , \tag{29}$$

corresponding to a convex parabolic height profile

$$h(xt) \propto -\frac{x^2}{2\lambda t} \ . \tag{30}$$

In Fig. 6 we have shown the ramp solution and the associated height profile.

### 3.3 Superposed Linear Modes

The character of the diffusive modes in the presence of a soliton is examined by means of a linear stability analysis of Eq. (18). Inserting

$$u = u_k + \delta u \ , \tag{31}$$

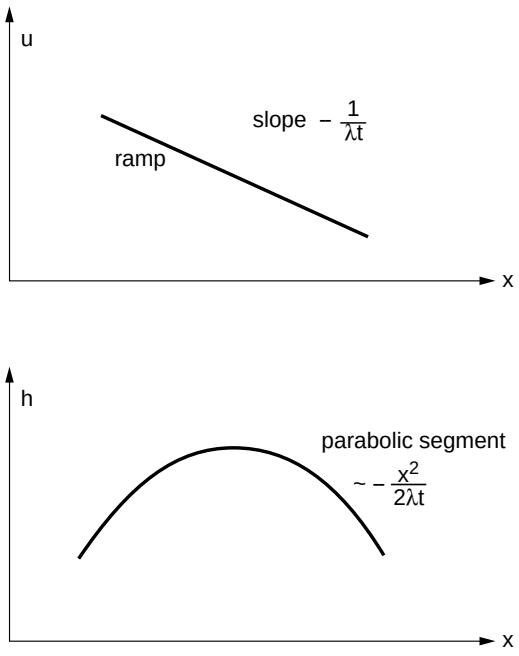


**Fig. 6.** We show a ramp solution for the noiseless Burgers equation together with the associated parabolic height profile.

and solving the associated linear eigenvalue problem we obtain, in addition to a localized translation mode, a band of damped diffusive modes

$$\delta u \propto \exp\left[-i\omega_k t + ikx\right] \ , \tag{32}$$

with a soliton-induced gap in the dispersion law

$$\omega_k = -i\nu(k^2 + k_s^2), \qquad k_s = \frac{\lambda u_+}{2\nu} \ . \tag{33}$$

In Fig. 7 we have shown the diffusive spectrum in the presence of the soliton.

The soliton modes and associated superposed diffusive modes are thus an essential feature of the nonlinear noiseless Burgers equation; this aspect will persist in the noisy case. The morphology of the noiseless Burgers equation is clear: The equation is basically damped but in the transient regime dissipative structures are present consisting of an initial value induced gas of "right hand" solitons connected by ramps with superposed linear diffusive modes. With vanishing slope field at the boundaries there is no deterministic current-input and

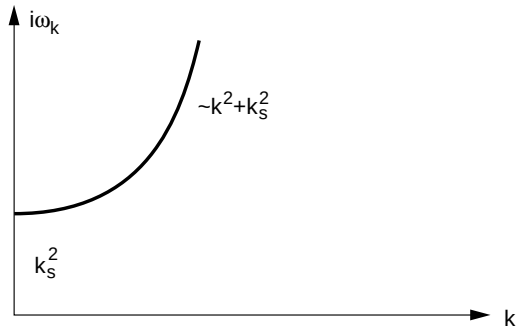


**Fig. 7.** We depict the diffusive dispersion law in the presence of a soliton for the noiseless Burgers equation. The gap is proportional to the soliton amplitude.

the morphology eventually decays through interaction and coalescence of moving solitons. In Fig. 8 we have shown the transient morphology of the noiseless Burgers equation and the associated height profile.

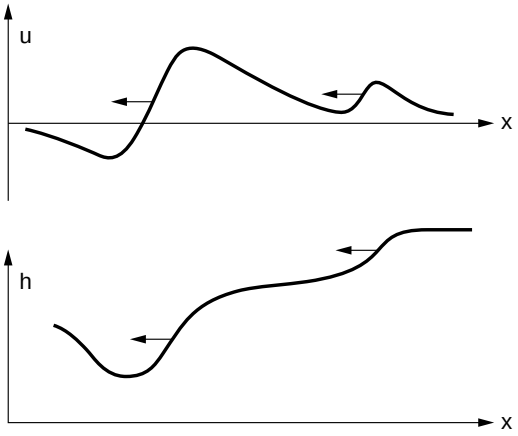


**Fig. 8.** We show the transient morphology for the noiseless Burgers equation consisting of "right hand" solitons connected by ramps for the slope field $u$ and downward cusps connected by parabolic segments for the associated height field $h$.

## 4 The Noisy Burgers Equation

In Sects. 2 and 3 we discussed the linear EW case in order to illustrate the singular character of the weak noise limit and the noiseless Burgers case in order to identify the essential nonlinear feature, i.e., the soliton or shock wave, in determining the morphology. Here we perform an analysis of the noisy Burgers equation (1) including both the nonlinear soliton modes and taking into account the singular character of the noise.

### 4.1 Phase Space Path Integral Formulation

The Martin-Siggia-Rose technique (Martin et al.(1973)) in a path integral formulation (Zinn-Justin (1989)) is the appropriate language to analyze the Burgers equation and works in the following way: All the information about the statistical properties of the Burgers equation is basically contained in the generator or characteristic function

$$Z(\mu) = \langle \exp\left[ i \int dxdt \mu u \right] \rangle \ . \tag{34}$$

The probability distribution $P(\{u(x)\}, t)$ and correlation functions are easily extracted from $Z$, e.g.,

$$\langle u(xt)u(x't') \rangle = -[\delta Z/\delta\mu(xt)\delta\mu(x't')]_{\mu=0} \ . \tag{35}$$

Here $\langle \cdots \rangle$ denotes an average withe respect to the noise $\eta$. In order to implement the Burgers equation (1) which provides the statistical link between the output slope field and the input noise driving the field, we insert the identity $\int \prod_{xt} du J \delta(\partial u/\partial t - \nu\nabla^2 u - \lambda u \nabla u - \nabla\eta) = 1$ in (34)(see note on the Jacobian $J$), expand the delta function, $\delta(x) \propto \int dp \exp(ipx)$, and average over the noise according to the Gaussian distribution in (2). This procedure yields the path integral

$$Z(\mu) \propto \int \prod_{xt} du dp \exp\left[ \frac{i}{\Delta} \tilde{S} \right] \exp\left[ i \int dxdt \mu u \right] \ , \tag{36}$$

where the action $\tilde{S}$ has the form

$$\tilde{S} = \int dxdt \left[ p \frac{\partial u}{\partial t} - \mathcal{H} \right] \ , \tag{37}$$

with Hamiltonian

$$\mathcal{H} = -\frac{i}{2}(\nabla p)^2 + p(\nu\nabla^2 u + \lambda u \nabla u) \ . \tag{38}$$

In the linear case for $\lambda = 0$ we expect the Hamiltonian to reduce to a harmonic form. This is accomplished by performing a shift operation of the noise variable $p$. Setting $p = \nu(iu - \varphi)$ we obtain the canonical phase space form

$$Z(\mu) \propto \int \prod_{xt} du d\varphi \exp\left[i\frac{\nu}{\Delta}S\right] \exp\left[i\frac{\nu}{\Delta}S_B\right] \exp\left[i\int dxdt\mu u\right] \ , \tag{39}$$

with bulk action

$$S = \int dxdt \left[u\frac{\partial\varphi}{\partial t} - \mathcal{H}\right] \ , \tag{40}$$

and canonical Hamiltonian

$$\mathcal{H} = -i\frac{\nu}{2}[(\nabla u)^2 + (\nabla\varphi)^2] + \frac{\lambda}{2}u^2\nabla\varphi \ . \tag{41}$$

The shift transformation also generates a surface term $S_B$ which contributes to the action for a particular slope configuration, depending on the choice of boundary condition. It will not play any role in the present discussion; see (Fogedby (1998)).

The Hamiltonian $\mathcal{H}$ is composed of a relaxational or irreversible harmonic component characterizing the EW case, and a nonlinear reversible mode coupling component associated with the drive $\lambda$ in the Burgers-KPZ case.

The path integral (39) has the usual Feynman form in phase space with $u$ and $\varphi$ as canonically conjugate variables, $u$ is the canonical "momentum" and the noise variable $\varphi$ the canonical "coordinate". We note, however, that $\mathcal{H}$ does not break up in a "kinetic energy" part and a "potential energy" part as is characteristic of an ordinary dynamical problem; the coordinate and momentum variables are mixed in the nonlinear interaction term. Furthermore, the Hamiltonian is complex characteristic of the Master equation for a relaxational problem.

Nevertheless, the Feynman form is very suggestive and allows us to draw both on "classical mechanics" in analyzing the path integral and on the associated "quantum field theory," yielding the path integral. We note that the noise strength $\Delta/\nu$ plays the role of an effective Planck constant defining a "correspondence principle" associated with the weak noise limit $\Delta \to 0$.

There are essentially two ways to approach the path integral (39): i) renormalized perturbation theory in the nonlinear term $\lambda$ and ii) a nonperturbative saddle point calculation for small $\Delta$.

Perturbation theory in $\lambda$ yields the same expansion that is obtained by directly expanding the noisy Burgers equation (1) and contracting the noise term by term according to (3). This expansion which is logarithmically divergent in the infrared limit in $d = 2$ and algebraically divergent below $d = 2$ requires a regularization scheme below $d = 2$ yielding renormalization group equations for $\nu$, $\lambda$, and $\Delta$. While $\lambda$ is invariant under scaling (yielding the scaling law in (10)), the effective expansion coefficient is $\lambda^2\Delta/\nu^3$ yielding the exponents

$(\zeta, z) = (1/2, 3/2)$ in $d = 1$. Note that the expansion is effectively in $\Delta$ and thus does not retrieve singular terms in $\Delta$.

As discussed earlier the noise $\Delta$ is the small parameter in the problem, at least in $d = 1$ where we have carried out the analysis. The singular terms in $\Delta$ are precisely obtained by performing a saddle point analysis of the path integral, corresponding to the WKB or "quasi-classical" limit of the Feynman path integral.

### 4.2 Field Equation for $\Delta \to 0$

In the weak noise limit $\Delta \to 0$ the path integral formulation allows for "a principle of least action" in that the leading saddle point or stationary contributions are obtained from the variational principle $\delta S = 0$ with respect to variations $\delta u$ and $\delta\varphi$. We thus obtain the Hamiltonian equations of motion

$$\frac{\partial u}{\partial t} = -\frac{\delta H}{\delta \varphi} \qquad \text{and} \qquad \frac{\partial \varphi}{\partial t} = +\frac{\delta H}{\delta u} \ , \tag{42}$$

where $H = \int dx \mathcal{H}$ and inserting (41) the field equations

$$\frac{\partial u}{\partial t} = -i\nu\nabla^2\varphi + \lambda u \nabla u \tag{43}$$

$$\frac{\partial \varphi}{\partial t} = +i\nu\nabla^2 u + \lambda u \nabla \varphi \ . \tag{44}$$

The coupled deterministic field equations (43-44) are an essential result of our analysis. In the weak noise limit $\Delta \to 0$ they effectively replace the noisy Burgers equation (1).

Note that we here have obtained a definite level of simplification in handling the statistical problem. The stochastic character of the Langevin equation (1) has been replaced by the path integral, yielding the additional variable $\varphi$ characterizing the noise, and in the weak noise limit the coupled field equations (43-44). The alternative formulation in terms of the deterministic Fokker-Planck equation is technically more difficult since the Fokker-Planck equation is a functional-differential equation, in fact "the Schrödinger equation" for the path integral. In Fig. 9 we have depicted the paths in $u\varphi$ phase space.

### 4.3 Linear Diffusive Modes

It is instructive first to discuss the linear EW case. For $\lambda = 0$ the field equations (43-44) reduce to the linear coupled pair

$$\frac{\partial u}{\partial t} = -i\nu\nabla^2\varphi \tag{45}$$

$$\frac{\partial \varphi}{\partial t} = +i\nu\nabla^2 u \ , \tag{46}$$

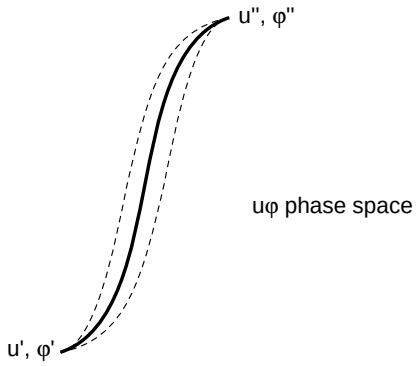


**Fig. 9.** We depict the paths in $u\varphi$ phase space connecting the initial values $u', \varphi'$ to the final values $u'', \varphi''$. The fully drawn extremal path corresponds to the stationary or saddle point solution for $\Delta \to 0$; the dashed paths characterize the fluctuations about the stationary solution.

admitting the solution

$$u(xt) \sim u_0^+ \exp\left[-i\omega_k^0 t + ikx\right] + u_0^- \exp\left[i\omega_k^0 t - ikx\right] \tag{47}$$

with gapless dispersion given by (24). We note the doubling of solutions, i.e., the equations imply both a damped and a growing solution, unlike the noiseless Burgers equation discussed in Sect. 3.1. This is a feature of the noisy case: In the stationary regime we attain time reversal invariance and both solutions are required in order to describe the stationary correlations.

For $\lambda = 0$ the path integral is Gaussian and we leave it as an exercise to derive the correlation function (15); it is also not difficult to derive the distribution (14).

### 4.4 Nonlinear Soliton Modes

Leaving aside the question whether the field equations (43-44) in the nonlinear case for $\lambda \neq 0$ are exactly integrable or admit a Cole-Hopf type transformation, we find, like in the case of the noiseless Burgers equation, permanent profile soliton solutions, see e.g. (Fogedby (1981)). In the static case we have

$$u_0(x) = \pm u_+ \tanh k_s(x - x_0) \tag{48}$$

of the same form as (25). Note, however, again a doubling of the solutions like in the linear case. In the stationary regime the noise excites both "right hand" and "left hand" solitons. Both modes are required in order to correctly describe the stationary growth morphology. In Fig.10 we have depicted the two soliton solutions and the associated height field.

**Fig. 10.** We depict the "right hand" and "left hand" solitons for the noisy Burgers equation and the associated upward and downward cusps in the associated height field.

### 4.5 Superposed Linear Modes

Like in the noiseless case we can perform a linear stability analysis of the field equations by inserting $u = u_0 + \delta u$ and $\varphi = \varphi_0 + \delta\varphi$. The resulting linear eigenvalue problem is exactly soluble, see e.g. (Fogedby et al. (1985)). In addition to a translation mode associated with a displacement of the soliton position $x_0$, we obtain a band of modes

$$\delta u \sim u^+ \exp\left[-i\omega_k t + ikx\right] + u^- \exp\left[i\omega_k t - ikx\right] \tag{49}$$

with a diffusive dispersion law with a gap given by Eq. (33). In Fig. 11 we have depicted a "right hand" soliton with a superposed diffusive mode.

### 4.6 A Growing Interface as a Dilute Soliton Gas

The saddle point modes, i.e., the solitons and diffusive modes, excited by the noise immediately allow a physical description of a growing interface. First we observe that the saddle point approximation actually implies a dilute gas of solitons connected by constant slope segments and satisfying the soliton condition (27); in the limit of well-separated solitons these configurations also provide saddle point solutions (Zinn-Justin (1989)). A single soliton corresponds to a downward or upward cusp in $h$. A pair of solitons describe a moving step, etc.

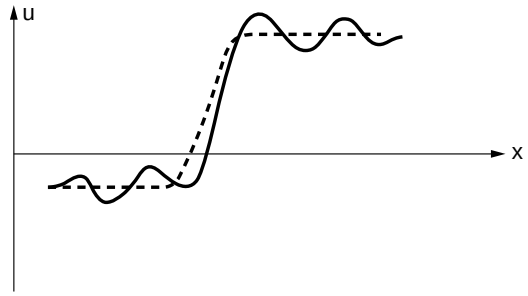


**Fig. 11.** We depict a "right hand" soliton (the dashed curve) with a superposed diffusive mode (the solid curve); the superposed damped linear mode is phase-shifted and exhibits a gap in the spectrum.

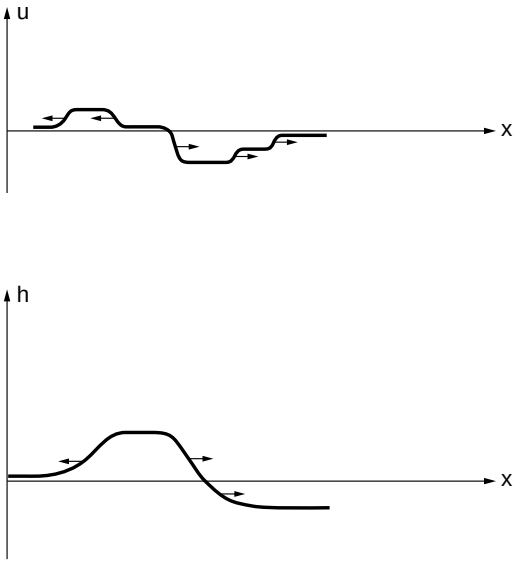


**Fig. 12.** We show the stationary growth morphology for the noisy Burgers equation, consisting of "right hand" and "left hand" solitons connected by horizontal segments for the slope field $u$ and downward and upward cusps connected by constant slope segments for the associated height field $h$.

In Fig. 12 we have depicted a general growth morphology in the slope field $u$ and the associated height field $h$. Superposed on the soliton configuration is a gas of diffusive modes (not indicated on the figure). We note that the stationary growth morphology is quite different from the transient morphology shown in Fig. 8.

### 4.7 Dynamics of Solitons

The Hamiltonian structure of the path integral allows for "a principle of least action" and we can associate dynamic and kinetic attributes to the soliton and diffusive modes. The energy is inferred from (41),

$$E = \int dx \left[ -i\frac{\nu}{2}[(\nabla u)^2 + (\nabla\varphi)^2] + \frac{\lambda}{2}u^2\nabla\varphi \right] \ . \tag{50}$$

The momentum, i.e., the generator of translation, has a form inferred from the Poisson bracket $\{u(x), \varphi(x')\} = \delta(x - x')$ for the canonically conjugate variable $u$ and $\varphi$,

$$P = \int dx u\nabla\varphi \ . \tag{51}$$

For a single soliton we have in terms of the boundary values

$$E = \ \pm i\frac{\lambda}{6}[u_+^3 - u_-^3] \tag{52}$$

$$P = \ \pm i\frac{1}{2}[u_+^2 - u_-^2] \ , \tag{53}$$

where $\pm$ indicates the "right hand" and "left hand" solitons, respectively.

In particular, for a two-soliton configuration representing a growing step with vanishing slope at the boundaries we obtain

$$E = \ \frac{8}{3}i\frac{|v|^3}{\lambda^2} \tag{54}$$

$$P = \ -4i\frac{v^2}{\lambda^2}sgn(v) \ . \tag{55}$$

We note the nonlinear velocity dependence characteristic of a soliton mode, see e.g. (Fogedby (1981)). Eliminating the velocity we infer the soliton dispersion law

$$E = i\lambda\frac{\sqrt{2}}{3}|P|^{3/2} \ , \tag{56}$$

and we recover the dynamic exponent $z = 3/2$ characterizing the gapless soliton dispersion. In Fig. 13 we have depicted the soliton dispersion law.

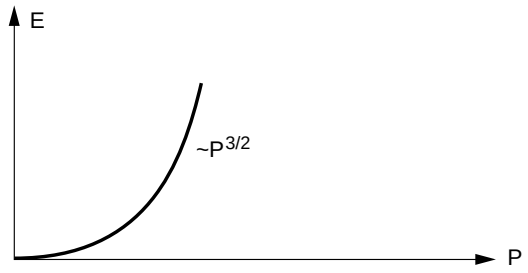


**Fig. 13.** We show the gapless soliton dispersion law characterized by the "fractional" exponent 3/2.

### 4.8 Stationary Distribution

As alluded to in Sect. 1 it is known that the stationary distribution in the Burgers-KPZ case is Gaussian and given by (11) (Huse, Henley, and Fisher (1985)). Here we attempt to make this result, which follows easily from the Fokker-Planck equation, plausible within the Martin-Siggia-Rose formalism.

From (36) it follows that the stationary distribution is given by

$$P_{\rm st}(u'') \propto \lim_{T\to 0} \int_{u'}^{u''} \prod_{xt} dpdu \exp\left[\frac{i}{\Delta}\tilde{S}\right] \ , \tag{57}$$

where $u' = u(x,0)$ and $u'' = u(x,T)$ are the initial and final values of the slope field and $\tilde{S}$ is integrated from $t = 0$ to $t = T$. In the weak noise limit $\Delta \to 0$ variation of $\tilde{S}$ yields saddle point equations equivalent to (43-44),

$$\frac{\partial u}{\partial t} = \nu\nabla^2 u + \lambda u\nabla u + i\nabla^2 p \tag{58}$$

$$\frac{\partial p}{\partial t} = -\nu\nabla^2 p + \lambda u\nabla p \ . \tag{59}$$

Using that the energy vanishes in the stationary state attained in the limit $T \to \infty$, we obtain from (37) at the saddle point

$$P_{\rm st}(u) \propto \lim_{T\to\infty} \left[\frac{i}{\Delta}\int^{T} dxdtp\frac{\partial u}{\partial t}\right] \ , \tag{60}$$

where $p$ and $u$ are solutions of the field equations (58-59). In order to demonstrate that $-ip \to 2\nu u$ for $T \to \infty$ implying the stationary distribution (11), $P_{\rm st} \propto \exp\left[-(\nu/\Delta)\int dx u^2\right]$, we define the deviation $\Delta u$ according to $-ip = 2\nu(u+\Delta u)$ and find to linear order

$$\left[\frac{\partial}{\partial t} - \lambda u\nabla\right]\Delta u = \nu\nabla^2\Delta u \ . \tag{61}$$

Owing to the Galilean invariance of the operator $\partial/\partial t - \lambda u\nabla$ we can choose an instantaneous frame with vanishing $u$ and (61) implies a decaying solution $\Delta u \propto \exp\left(-\nu k^2 t\right)$. This is not a rigorous proof; basically we assume that the trajectories on the $E = 0$ energy surface in $u\varphi$ phase space at long times are attracted to the sub manifold defined by $-ip = 2\nu u$.

## 5 Scaling and Universality Classes

The present path integral approach also permits a simple discussion of the scaling properties of the noisy Burgers equation. Focussing on the slope correlation function (39) yields

$$\langle u(xt)u(x't')\rangle \propto \int\prod dud\varphi \exp\left[i(\nu/\Delta)S\right]u(xt)u(x't') \ , \tag{62}$$

where we assuming vanishing boundary conditions have set $S_B = 0$. The direct evaluation of the path integral (62) requires the application of methods from quantum chaos such as periodic orbit theory, see e.g. (Dashen et al. (1974)) and is still in progress. However, by discussing $\langle uu\rangle$ in terms of the underlying "quantum field theory" it is an easy task to extract the scaling properties.

In the "quantum description" the canonical fields $u$ and $\varphi$ are replaced by "quantum operators" $\hat{u}$ and $\hat{\varphi}$ and likewise the Hamiltonian and momentum in (41) and (51). The correlation function (62) can thus be expressed as the time-ordered product (Zinn-Justin (1989))

$$\langle u(xt)u(00)\rangle \propto \langle 0|T\hat{u}(xt)\hat{u}(00)|0\rangle \ , \tag{63}$$

where $\hat{u}$ evolves in time according to the "quantum Hamiltonian" (41) and $|0\rangle$ denotes the zero-energy stationary state. Displacing the field from $(x,t)$ to $(0,0)$, using the Hamiltonian and momentum operators, we have

$$\hat{u}(xt) = \exp\left[i(\nu/\Delta)(\hat{P}x + \hat{H}t)\right]\hat{u}(00)\exp\left[-i(\nu/\Delta)(\hat{P}x + \hat{H}t)\right] \ , \tag{64}$$

which inserted in (63) together with a complete set of intermediate quasi-particle wavenumber states $|K\rangle$, $P = (\Delta/\nu)K$, with frequency $\Omega$, $E = (\Delta/\nu)\Omega$, yields the spectral representation

$$\langle u(xt)u(00)\rangle \propto \int dKG(K)\exp\left[-i(\Omega t - Kx)\right] \ . \tag{65}$$

Here $G(K)$ is a form factor and $\Omega, K$ are the frequencies and wavenumbers of the quasi-particles in the theory.

The scaling limit for large $x$ and large $t$ corresponds to the bottom of the quasi-particle spectrum and we note that only gapless excitations contribute. Assuming a general dispersion law with exponent $\beta$,

$$\Omega = AK^{\beta} \ , \tag{66}$$

and assuming that the form factor $G(K)$ is regular for small wavenumber, $G(K) \propto$ const., we obtain, rescaling $K$

$$\langle u(xt)u(00)\rangle \propto x^{-1} \int dK e^{-iAK^{\beta}(t/x^{\beta})-iK} \ . \tag{67}$$

Comparing the spectral form (67) with the dynamic scaling hypothesis (7) we first infer the robust roughness exponent $\zeta = 1/2$, independent of the quasi-particle dispersion law. The dynamic exponent $z$ is given by the exponent $\beta$ for the quasi-particle dispersion law. In the linear EW case the gapless diffusive dispersion law $\Omega \propto k^2$ yields the dynamic exponent $z = 2$; in the Burgers-KPZ case the diffusive modes develop a gap and do not contribute to the scaling; however, the noise excites a new nonlinear gapless soliton mode with dispersion $\Omega \propto K^{3/2}$, yielding the exponent $z = 3/2$. We also obtain a heuristic expression for the scaling function $F_3$ in (7),

$$F_3(w) \propto \int dK e^{-i(K^z w+K)} \ , \tag{68}$$

which has the same form as the probability distribution for Lévy flights (Fogedby (1994)). In Table 1 we have summarized the exponents and universality classes for the EW and Burgers-KPZ cases.

**Table 1.** Scaling exponents and universality classes

| Model | Roughness exp. ($\zeta$) | Dynamic exp. ($z$) | Universality class |
|---|---|---|---|
| EW equation | $\frac{1}{2}$ | 2 | EW |
| Burgers equation | $\frac{1}{2}$ | $\frac{3}{2}$ | Burgers-KPZ |

## 6 Summary and conclusion

We have here advanced a novel approach to the growth morphology and scaling behavior of the noisy Burgers equation in one dimension. Using the Martin-Siggia-Rose (MSR) technique in a canonical form we have demonstrated that

the physics of the strong coupling fixed point is associated with an essential singularity in the noise strength and can be accessed by appropriate theoretical soliton techniques.

The canonical representation of the MSR functional integral in terms of a Feynman phase space path integral with a complex Hamiltonian identifies the noise strength as the relevant small nonperturbative parameter and allows for *a principle of least action*. In the asymptotic weak noise limit the leading contributions to the path integral are given by a dilute gas of solitons with superposed linear diffusive modes. The canonical variables are the local slope of the interface and an associated "conjugate" noise field, characteristic of the MSR formalism. In terms of the local slope the soliton and diffusive mode picture provides a many-body description of a growing interface governed by the noisy Burgers equation. The noise-induced slope fluctuations are here represented by the various paths or configurations contributing to the path integral. Moreover, a spectral representation of the slope correlations based on the underlying "quantum field theory" gives access to the scaling exponents and provide a heuristic expression for the scaling function.

So far the present nonperturbative approach has only been implemented in the one dimensional case where the analysis is tractable. However, the singular nature of the weak noise limit seems to be a general feature of the onset of the stationary regime and might also be important for the Burgers-KPZ equations in higher dimension. It remains to be investigated whether the present saddle point approach can be generalized to this case.

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